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On a Problem of T. J. Rivlin in Approximation Theory

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INTRODUCTION

T. J. Rivlin has recently raised the following problem [5]: Characterize those n-tuples of algebraic polynomials $\{p_0, p_1, \ldots, p_{n-1}\}$, with degrees satisfying

$$\deg p_j = j \qquad (j = 0, 1, \ldots, n-1), \tag{1}$$

for which there exists an $x \in C([0,1])$ such that the polynomial of best approximation of degree j to x (in the sense of Čebyšev) is p_j $(j = 0, 1, \ldots, n-1)$. What is the answer in the particular case $n = 2$?

In the present paper we shall consider the following more general problem: Let $\{G_k\}$ be a sequence of linear subspaces of a normed linear space E. Characterize those sequences $\{g_k\}$ in E, with $g_k \in G_k$ $(k = 1, 2, \ldots)$, for which there exists an $x \in E$ such that

$$g_k \in \mathscr{P}_{G_k}(x) \qquad (k = 1, 2, \ldots) \tag{2}$$

where $\mathscr{P}_G(x)$ denotes the set of all elements of best approximation to x from G, i.e., the set

$$\{g_0 \in G \mid \|x - g_0\| = \inf_{g \in G} \|x - g\|\}.$$

We shall devote most of our attention to the cases when

$$G_1 \subset G_2 \subset \ldots, \tag{3}$$

or

$$G_1 \supset G_2 \supset \ldots, \tag{4}$$

both in general and in some concrete normed linear spaces; in particular, we shall give a complete answer to the second question of T. J. Rivlin in $C([0,1])$. Finally, we shall also consider the problem of characterizing the sequences of subspaces $\{G_k\}$ satisfying (3) or (4) and with the following property, which we

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shall call property (A): for every sequence $\{g_k\}$ with $g_k \in G_k$ $(k=1,2,\ldots)$ there exists an $x \in E$ satisfying (2).

Let us mention that the above problems are somewhat analogous to "the inverse problem of the theory of best approximation" (i.e., the problem of finding an $x \in E$ with prescribed "best error" values $e_{G_k}(x) = \inf_{g \in G_k} \|x - g\|$, $(k=1,2,\ldots)$ raised by S. Bernstein [2].

We conclude the introduction with a brief review of some useful notation and terminology. (All our notation conforms to that of the monograph [6]). We recall that $\pi_G^{-1}(g)$, where $g \in G$, denotes the set of all $y \in E$ such that $g \in \mathscr{P}_G(y)$. A subspace G of E is called a *Čebyšev subspace* if each $x \in E$ has a unique element of best approximation $g_0 \in G$. In this case the mapping $\pi_G\colon x \to g_0$ is called the *metric projection* of E onto G.

1. Some Results in General Normed Linear Spaces

A solution of the main problem of the Introduction is given by

Theorem 1. *Let $\{G_k\}$ be a sequence of linear subspaces of a normed linear space E, and let $g_k \in G_k$ $(k=1,2,\ldots)$. There exists an $x \in E$ satisfying* (2) *if and only if there exist elements $y_k \in \pi_{G_k}^{-1}(0)$ $(k=1,2,\ldots)$ such that*

$$g_{k+1} - g_k = y_{k+1} - y_k \qquad (k=1,2,\ldots). \tag{5}$$

Proof. Assume that there exists an $x \in E$ satisfying (2). Put

$$y_k = g_k - x \qquad (k=1,2,\ldots). \tag{6}$$

Then by (2) we have $0 \in \mathscr{P}_{G_k}(y_k)$, i.e., $y_k \in \pi_{G_k}^{-1}(0)$ $(k=1,2,\ldots)$, and by consecutive subtraction in (6) we obtain (5).

Conversely, assume that there exist elements $y_k \in \pi_{G_k}^{-1}(0)$ such that we have (5). Put

$$x = g_1 - y_1. \tag{7}$$

Then by (5) we have

$$x = g_1 - y_1 = g_2 - y_2 = \cdots \tag{8}$$

whence, by $0 \in \mathscr{P}_{G_k}(y_k)$, we obtain

$$\|x - g_k\| = \|y_k\| \leqslant \|y_k - g_k + g\| = \|x - g\| \qquad (g \in G_k, k=1,2,\ldots),$$

i.e., (2), which completes the proof.

Theorem 2. *Let $\{G_k\}$ be a sequence of Čebyšev subspaces of a normed linear space E, satisfying* (3), *such that the metric projections π_{G_k} $(k=1,2,\ldots)$ are linear, and let $g_k \in G_k$ $(k=1,2,\ldots)$. In order that there exist an $x \in E$ satisfying*

$$\pi_{G_k}(x) = g_k \quad (k=1,2,\ldots), \tag{9}$$

it is necessary, and if $\overline{\bigcup_{i=1}^{\infty} G_i}$ *is reflexive or if* $G_n = G_{n+1} = G_{n+2} = \ldots$, *it is sufficient that*

$$\sup_k \|g_k\| < \infty, \tag{10}$$

$$g_{k+1} - g_k \in \pi_{G_k}^{-1}(0) \qquad (k = 1, 2, \ldots). \tag{11}$$

Proof. Assume that there exists an $x \in E$ satisfying (9). Then

$$\|g_k\| = \|\pi_{G_k}(x)\| \leqslant \|\pi_{G_k}(x) - x\| + \|x\| \leqslant 2\|x\| \qquad (k = 1, 2, \ldots),$$

whence we infer (10). Furthermore, by Theorem 1 we have

$$g_{k+1} - g_k \in \pi_{G_{k+1}}^{-1}(0) - \pi_{G_k}^{-1}(0) \qquad (k = 1, 2, \ldots), \tag{12}$$

whence, since now each $\pi_{G_k}^{-1}(0)$ is a linear subspace of E because π_{G_k} is linear (see [6], Ch. I, Theorem 6.4), and since, by (3), $\pi_{G_{k+1}}^{-1}(0) \subset \pi_{G_k}^{-1}(0)$ $(k = 1, 2, \ldots)$, we infer (11).

Conversely, assume that we have (10) and (11). Fix arbitrary n, k, with $1 \leqslant k \leqslant n - 1$, and put

$$y_k = g_n - g_k. \tag{13}$$

Then, since each $\pi_{G_k}^{-1}(0)$ is a linear subspace of E, and since, by (3), $\pi_{G_{l+1}}^{-1}(0) \subset \pi_{G_l}^{-1}(0)$ $(l = 1, 2, \ldots)$, we obtain from (11)

$$\begin{aligned} y_k &= (g_n - g_{n-1}) + (g_{n-1} - g_{n-2}) + \ldots + (g_{k+1} - g_k) \\ &\in \pi_{G_{n-1}}^{-1}(0) + \pi_{G_{n-2}}^{-1}(0) + \ldots + \pi_{G_k}^{-1}(0) = \pi_{G_k}^{-1}(0), \end{aligned}$$

whence, by the quasi-additivity of π_{G_k} (see e.g. [6], Ch. I, Theorem 6.1),

$$\pi_{G_k}(g_n) = \pi_{G_k}(y_k + g_k) = \pi_{G_k}(y_k) + g_k = g_k.$$

Since n, k, with $1 \leqslant k \leqslant n - 1$, were arbitrary, and since $\pi_{G_n}(g_n) = g_n$, it follows that

$$\pi_{G_l}(g_{l+m}) = g_l \qquad (l, m = 1, 2, \ldots). \tag{14}$$

Now, assuming that $\overline{\bigcup_{i=1}^{\infty} G_i}$ is reflexive, there exists by (10) a subsequence $\{g_{k_j}\}$ of $\{g_k\}$, converging weakly to an element $x \in E$. Since π_{G_l} is linear, it is continuous on E, whence also weakly continuous, and hence, taking into account (14), we infer

$$g_l = w - \lim_{j \to \infty} \pi_{G_l}(g_{k_j}) = \pi_{G_l}(x) \qquad (l = 1, 2, \ldots),$$

i.e., (9). On the other hand, assuming that $G_n = G_{n+1} = G_{n+2} = \ldots$, by (11) we must have $g_{n+1} - g_n = 0$, $g_{n+2} - g_{n+1} = 0, \ldots$. Consequently, putting

$$x = g_n (= g_{n+1} = g_{n+2} = \ldots),$$

by (14), we have again (9), which completes the proof of Theorem 2.

Remark 1. One can also give other equivalent formulations of condition (11), e.g., the following ones:

$$g_{k+1} \in \pi_{G_k}^{-1}(g_k) \qquad (k, = 1, 2, \ldots), \tag{11a}$$

$$g_{k+1} - g_k \perp G_k \qquad (k = 1, 2, \ldots), \tag{11b}$$

where $x \perp y$ if and only if $\|x + \alpha y\| \geqslant \|x\|$ for all scalars α, and $x \perp G$ if and only if $x \perp g$ for all $g \in G$.

Remark 2. In the sufficiency part of Theorem 2 some additional assumption (like the reflexivity of $\overline{\bigcup_{i=1}^{\infty} G_i}$) is indeed necessary, as shown by the following example: Let $E = c_0$ endowed with the norm

$$|x| = \sup_j |\xi_j| + \sum_{i=1}^{\infty} \frac{1}{2^i} |\xi_i| \qquad (x = \{\xi_n\} \in c_0), \tag{15}$$

and for each k, let G_k be the linear subspace $[e_1, \ldots, e_k]$ of E spanned by $\{e_1, \ldots, e_k\}$, where e_j is the jth unit vector $\{\underbrace{0, \ldots, 0}_{j-1}, 1, 0, \ldots\}$. Then $|\ |$ is equivalent to the initial norm on c_0, and it is a T-norm (see [3], [7]) with respect to the unit vector basis $\{e_n\}$ of E, i.e., each G_k is a Čebyšev subspace of E and

$$\pi_{G_k}(x) = \{\xi_1, \ldots, \xi_k, 0, 0, \ldots\} \qquad (x = \{\xi_n\} \in E), \tag{16}$$

whence each π_{G_k} is linear. However, $\overline{\bigcup_{i=1}^{\infty} G_i} = E$ is nonreflexive, and for

$$g_k = \sum_{i=1}^{k} e_i = \{\underbrace{1, \ldots, 1}_{k}, 0, 0, \ldots\} \qquad (k = 1, 2, \ldots) \tag{17}$$

there exists no $x \in E$ satisfying (9) (since by (16) the only possible such x is $\{1, 1, \ldots\} \notin c_0$), although this sequence $\{g_k\}$ satisfies conditions (10) and (11) of Theorem 2 (since by (16) $\pi_{G_k}(g_{k+1} - g_k) = \pi_{G_k}(e_{k+1}) = 0$).

Remark 3. If $G_n = G_{n+1} = G_{n+2} = \ldots$, then, obviously, condition (10) in Theorem 2 can be omitted. However, if we only assume that $\overline{\bigcup_{i=1}^{\infty} G_i}$ is reflexive, this is no longer the case, as shown by the following example: Let $E = l^2$, $G_k =$ the subspace $[e_1, \ldots, e_k]$ of E spanned by $\{e_1, \ldots, e_k\}$, $(k = 1, 2, \ldots)$, where $\{e_n\}$ is the unit vector basis of E, and

$$g_k = \sum_{i=1}^{k} e_i = \{\underbrace{1, \ldots, 1}_{k}, 0, 0, \ldots\} \qquad (k = 1, 2, \ldots). \tag{18}$$

Then, again, the norm in E is a T-norm with respect to $\{e_n\}$, i.e., each G_k is a Čebyšev subspace of E, and we have (16); whence each π_{G_k} is linear. Furthermore, $\overline{\bigcup_{i=1}^{\infty} G_i} = E$ is reflexive. However, $\sup_k \|g_k\| = \infty$, and there exists no $x \in E$ satisfying (9) (since, by (16), the only possible such x is $\{1, 1, \ldots\} \notin l^2$), although $\{g_k\}$ satisfies (11) (since, for every $g = \sum_{i=1}^{k} \alpha_i e_i \in G_k$, we have

$$\|g_{k+1} - g_k\| = \|e_{k+1}\| = 1 \leqslant \left(1 + \sum_{i=1}^{k} |\alpha_i|^2\right)^{1/2} = \|g_{k+1} - g_k - g\|).$$

THEOREM 3. *Let $\{G_k\}$ be a sequence of Čebyšev subspaces of a normed linear space E, satisfying* (4), *and such that the metric projections π_{G_k} are linear. Let $g_k \in G_k$ $(k = 1, 2, \ldots)$. There exists an $x \in E$ satisfying* (9) *if and only if*

$$g_{k+1} - g_k \in \pi_{G_{k+1}}^{-1}(0) \qquad (k = 1, 2, \ldots). \tag{19}$$

Proof. Assume that there exists an $x \in E$ satisfying (9). Then, by Theorem 1, we have (12), whence, since now each $\pi_{G_k}^{-1}(0)$ is a linear subspace of E (because π_{G_k} is linear), and since, by (4), $\pi_{G_k}^{-1}(0) \subset \pi_{G_{k+1}}^{-1}(0)$ $(k = 1, 2, \ldots)$, we infer (19).

Conversely, assume that we have (19). Put

$$x = g_1. \tag{20}$$

Then, since each $\pi_{G_k}^{-1}(0)$ is a linear subspace of E, and since, by (4), $\pi_{G_k}^{-1}(0) \subset \pi_{G_{k+1}}^{-1}(0)$ $(k = 1, 2, \ldots)$, we obtain from (19)

$$\begin{aligned} g_n - x &= (g_n - g_{n-1}) + (g_{n-1} - g_{n-2}) + \ldots + (g_2 - g_1) \\ &\in \pi_{G_n}^{-1}(0) + \pi_{G_{n-1}}^{-1}(0) + \ldots + \pi_{G_2}^{-1}(0) = \pi_{G_n}^{-1}(0) \quad (n = 2, 3, \ldots), \end{aligned}$$

whence, by the quasi-additivity of π_{G_n}, it follows

$$g_n - \pi_{G_n}(x) = \pi_{G_n}(g_n - x) = 0 \qquad (n = 2, 3, \ldots),$$

i.e. (9) (since obviously $\pi_{G_1}(x) = \pi_{G_1}(g_1) = g_1$), which completes the proof of Theorem 3.

Comparing Theorems 2 and 3, we see that the situation for decreasing sequences of subspaces is "better", since we need not make any additional assumption like the reflexivity of $\overline{\bigcup_{i=1}^{\infty} G_i}$, and since the condition of boundedness of $\{g_k\}$ can be omitted.

Let us consider now property (A) (see the Introduction). A sequence of subspaces $\{G_k\}$ is called *nontrivial* if, for some index $k > 1$, $G_k \neq \{0\}$.

THEOREM 4. *Let* $\{G_k\}$ *be a nontrivial sequence of linear subspaces of a normed linear space E, satisfying* (3) *or* (4), *and such that at least one* $G_k \neq \{0\}$ *is a Čebyšev subspace. Then* $\{G_k\}$ *does not have property* (A).

Proof. Assume that $G_k \neq \{0\}$ is a Čebyšev subspace, and that there exists an index l such that $G_k \subset G_l$. Take $g_k \in G_k$, and $g_l \in G_k \backslash \{g_k\}$. If there existed an $x \in E$ satisfying (2), then we would have $g_l \in \mathscr{P}_{G_k}(x) \cap G_k \subset \mathscr{P}_{G_k}(x)$, contradicting the assumption that G_k is Čebyšev.

Assume now that $G_k \neq \{0\}$ is a Čebyšev subspace, and that there exists no index l such that $G_k \subset G_l$. Since $\{G_k\}$ is nontrivial, there exists an index l with $G_k \supset G_l \neq \{0\}$. Take $g_l \in G_l$, and $g_k \in G_l \backslash \{g_l\}$. If there existed an $x \in E$ satisfying (2), then, since G_k is Čebyšev, we would have $\|x - g_k\| < \|x - g\|$ for all $g \in G_k \backslash \{g_k\}$, whence, in particular, $\|x - g_k\| < \|x - g_l\|$, which contradicts the fact that $g_l \in \mathscr{P}_{G_l}(x)$, since $g_k \in G_l$. This completes the proof.

Remark 4. The only case excluded by the hypothesis that $\{G_k\}$ be nontrivial is the case when $G_k = \{0\}$ for all $k > 1$. In this case it is easily seen that property (A) always holds.

2. THE CASE OF HILBERT SPACES

THEOREM 5. *Let* $\{G_k\}$ *be a sequence of closed linear subspaces of a Hilbert space E, satisfying* (3), *and let* $g_k \in G_k$ $(k = 1, 2, \ldots)$. *There exists an* $x \in E$ *satisfying* (9), *if and only if*

$$\sup_k \|g_k\| < \infty, \tag{21}$$

$$(g_{k+1} - g_k, g) = 0 \qquad (g \in G_k; k = 1, 2, \ldots). \tag{22}$$

Moreover, in this case, the sequence $\{g_k\}$ *converges* (*in the norm-topology*), *and we have*

$$\lim_{k\to\infty} g_k = \pi_G(x), \tag{23}$$

where $G = \overline{\bigcup_{i=1}^{\infty} G_i}$, *and where x is any element in E satisfying* (9).

Proof. The first part is an immediate consequence of Theorem 2 and Remark 1.

Assume now that x is an arbitrary element in E satisfying (9). We claim that, in this case, $\{g_k\}$ is a "minimizing sequence" for x in the subspace G, i.e.,

$$\lim_{k\to\infty} \|x - g_k\| = \inf_{g \in G} \|x - g\|. \tag{24}$$

Indeed, let $\epsilon > 0$ be arbitrary, and let $g' \in G$ be such that $\|x - g'\| \leqslant \inf_{g\in G} \|x - g\| + \epsilon/2$. Then, by the definition of G, there exist an index $N = N(\epsilon)$ and an element $g'' \in G_N \subset G_{N+1} \subset \ldots$ such that $\|g' - g''\| < \epsilon/2$. Consequently, by (9),

$$\inf_{g \in G} \|x - g\| \leqslant \|x - g_k\| = \|x - \pi_{G_k}(x)\| \leqslant \|x - g''\|$$
$$\leqslant \|x - g'\| + \|g' - g''\| \leqslant \inf_{g \in G} \|x - g\| + \epsilon \qquad (k \geqslant N(\epsilon)),$$

whence, since $\epsilon > 0$ was arbitrary, we infer (24).

However, (24) implies (see, e.g., [4], p. 248, Lemma 2) that the sequence $\{g_k\}$ converges (in the norm-topology) to an element $g_0 \in G$, whence

$$\lim_{k \to \infty} \|x - g_k\| = \|x - g_0\|.$$

Consequently, taking into account (24), we have $\|x - g_0\| = \inf_{g \in G} \|x - g\|$, i.e., $g_0 = \pi_G(x)$, which completes the proof of Theorem 5.

Remark 5. From the above it follows that for any pair $x', x'' \in E$ satisfying (9), we have $\pi_G(x') = \pi_G(x'') = \lim_{k \to \infty} g_k$. On the other hand, the proof of Theorem 2 shows that $x = w - \lim_{k \to \infty} g_k = \lim_{k \to \infty} g_k \in G$ itself also satisfies (9) (for this particular x we have, of course, $\pi_G(x) = x$).

THEOREM 6. *Let $\{G_k\}$ be a sequence of closed linear subspaces of a Hilbert space E, satisfying* (4), *and let $g_k \in G_k$ $(k = 1, 2, \ldots)$. There exists an $x \in E$ satisfying* (9), *if and only if*

$$(g_{k+1} - g_k, g) = 0 \qquad (g \in G_{k+1}; k = 1, 2, \ldots). \tag{25}$$

Moreover, in this case, the sequence $\{g_k\}$ converges (in the norm-topology), and we have

$$\lim_{k \to \infty} g_k = \pi_G(x), \tag{26}$$

where $G = \bigcap_{i=1}^{\infty} G_i$, and where x is any element in E satisfying (9).

Proof. The first part is an immediate consequence of Theorem 3.

Now let x be an arbitrary element in E satisfying (9). We shall prove that $\{g_k\}$ is a minimizing sequence for x in G, i.e., that we have (24).

Observe, first, that $\lim_{k \to \infty} \|x - g_k\|$ exists and is $\leqslant \inf_{g \in G} \|x - g\|$, since

$$\|x - g_k\| = \inf_{g \in G_k} \|x - g\| \leqslant \inf_{g \in G_{k+1}} \|x - g\| = \|x - g_{k+1}\| \qquad (k = 1, 2, \ldots),$$

and since

$$\|x - g_k\| = \inf_{g \in G_k} \|x - g\| \leqslant \inf_{g \in G} \|x - g\| \qquad (k = 1, 2, \ldots)$$

(by virtue of $G_k \supset G_{k+1} \supset G$).

Since, by (9), we have $\sup_k\|g_k\| \leqslant 2\|x\|$, we can extract a subsequence $\{g_{k_j}\}$ of $\{g_k\}$, converging weakly to an element $y \in E$. Then we have

$$\|x-y\|\,\|x-g_{k_j}\| \geqslant |(x-y, x-g_{k_j})| \to |(x-y, x-y)| = \|x-y\|^2, \qquad \text{as } j \to \infty. \tag{27}$$

Furthermore, since $g_l \in G_i$ for $l = i, i+1, i+2, \ldots$ $(i = 1,2,\ldots)$, we have $y \in \bigcap_{i=1}^{\infty} G_i = G$. Now, if $x = y$, then $x \in G = \bigcap_{i=1}^{\infty} G_i$, whence $g_k = \pi_{G_k}(x) = x$ $(k = 1,2,\ldots)$, and, thus, (24) holds (with 0 on both sides). On the other hand, if $x \neq y$, then from (27) we obtain

$$\lim_{j\to\infty} \|x-g_{k_j}\| \geqslant \|x-y\| \geqslant \inf_{g\in G}\|x-g\| \geqslant \lim_{j\to\infty}\|x-g_{k_j}\|,$$

whence $\lim_{j\to\infty}\|x-g_{k_j}\| = \inf_{g\in G}\|x-g\| = \|x-y\|$, i.e. $\{g_{k_j}\}$ is a minimizing sequence, and $y = \pi_G(x)$.

Now, if $\{g_k\}$ itself were not a minimizing sequence, there would exist an $\epsilon_0 > 0$ and an infinite sequence of indices, say $\{i_n\}$, such that

$$\|x-g_{i_n}\| - \inf_{g\in G}\|x-g\| \geqslant \epsilon_0 \qquad (n = 1,2,\ldots), \tag{28}$$

and then, repeating the above argument for $\{g_{i_n}\}$ instead of $\{g_k\}$, we would obtain a minimizing subsequence of $\{g_{i_n}\}$, contradicting (28). Thus, the sequence $\{g_k\}$ itself is a minimizing sequence.

Consequently, as in the final part of the proof of Theorem 4, $\{g_l\}$ is convergent (in the norm-topology) to an element $g_0 \in G$, which, by (24), must coincide with $\pi_G(x)$ (this also follows from $w - \lim_{j\to\infty} g_{k_j} = y = \pi_G(x)$). This completes the proof of Theorem 6.

Finally, from Theorem 4 it follows that *no nontrivial sequence of closed linear subspaces of a Hilbert space E, satisfying* (3) *or* (4), *has property* (A).

3. Some Results in Spaces of Continuous Functions

The answer to the second question of T. J. Rivlin (see the Introduction) is given by

THEOREM 7. *Let* $E = C([0,1])$, *let* G_1, G_2 *be the Čebyšev subspaces* $G_1 = [z_1]$, $G_2 = [z_1, z_2]$, *where* $z_1(t) \equiv 1$, $z_2(t) \equiv t$ $(t \in [0,1])$, *and let* $g_k \in G_k$ $(k = 1,2)$. *There exists an* $x \in E$ *satisfying*

$$\pi_{G_k}(x) = g_k \qquad (k = 1,2) \tag{29}$$

if and only if the linear function $g = g_2 - g_1$ *is either* $\equiv 0$ *or has one change of sign in* $[0,1]$.

Proof. Assume that there exists an $x \in E$ satisfying (29), but the condition of the theorem is not satisfied, i.e., $g \not\equiv 0$ and does not change sign in $[0,1]$, say $g \geqslant 0$ on $[0,1]$.

By virtue of Theorem 1 and the alternation theorem of Čebyšev, there exist elements

$$y_1 \in \pi_{G_1}^{-1}(0) = \{y \in C([0,1]) \mid \text{there exist } t_1 < t_2 \text{ in } [0,1] \text{ with } y(t_1) = -y(t_2) = \delta \|y\|\}, \tag{30}$$

$$y_2 \in \pi_{G_2}^{-1}(0) = \{y \in C([0,1]) \mid \text{there exists } t_3 < t_4 < t_5 \text{ in } [0,1] \text{ with } y(t_3) = -y(t_4) = y(t_5) = \delta' \|y\|\}, \tag{31}$$

where $\delta, \delta' = \pm 1$, such that

$$g = g_2 - g_1 = y_2 - y_1. \tag{32}$$

We claim that

$$\|y_1\| = \|y_2\|. \tag{33}$$

Indeed, if this claim were not true, then, since $y_2 = g + y_1 \geqslant y_1$, we would have $\|y_2\| > \|y_1\|$, whence

$$- y_2\| < -\|y_1\| \leqslant y_1(t) \leqslant y_2(t) \qquad (t \in [0,1]),$$

contradicting (31).

Now from (33) and (32) it follows that $g(t_0) = 0$ for each $t_0 \in [0,1]$ such that $y_1(t_0) = \|y_1\|$. Since the linear function g has at most one zero, it follows that we have $g(t) > 0$ for all $t \neq t_0$, whence

$$y_2(t) \equiv g(t) + y_1(t) > y_1(t) \geqslant -\|y_1\| = -\|y_2\| \qquad (t \in [0,1] \setminus \{t_0\})$$

which, since $y_2(t_0) = y_1(t_0) = \|y_1\| = \|y_2\|$, contradicts (31).

In the case when $g \leqslant 0$ on $[0,1]$, we arrive at a contradiction by a similar argument. This proves that the condition is necessary.

Assume now that the condition is satisfied, i.e., $g = g_2 - g_1$ is either $\equiv 0$, or has one change of sign in $[0,1]$.

Then, if $g \equiv 0$, for $x = g_1 = g_2$ we obviously have (29).

On the other hand, if

$$g(t) \equiv g_2(t) - g_1(t) \equiv at + b \tag{34}$$

has one change of sign in $[0,1]$, then $a \neq 0$ and $0 < -b/a < 1$. We have to consider several cases:

Case 1. $a > 0$, and $0 < -b/a < \frac{1}{2}$. Put

$$y_1(t) = \begin{cases} \|g\| & \text{for } t = 0 \\ -\|g\| & \text{for } t = -2\dfrac{b}{a} \\ \|g\| - a & \text{for } t = 1 \\ \text{linear for the other } t. \end{cases} \tag{35}$$

Then, since $0 < a = g(1) - g(0) \leqslant 2\|g\|$, we have $|\,\|g\| - a| \leqslant \|g\|$, whence (30) with $t_1 = 0$, $t_2 = -2b/a$, $\delta = 1$. Furthermore, for the function

$$y_2(t) = g(t) + y_1(t) = \begin{cases} \|g\| + b & \text{for } t = 0 \\ -\|g\| - b & \text{for } t = -2\dfrac{b}{a} \\ \|g\| + b & \text{for } t = 1 \\ \text{linear for the other } t \end{cases} \tag{36}$$

we have (32) and (31) with $t_3 = 0$, $t_4 = -2b/a$, $t_5 = 1$, $\delta' = 1$, whence, by Theorem 1, there exists an $x \in C([0,1])$ satisfying (29).

Case 2. $a < 0$ and $0 < -b/a < \frac{1}{2}$. Then $-g = -at - b$, with $-a > 0$, whence, by case 1 above, $-g = y_2 - y_1$ with $y_k \in \pi_{G_k}^{-1}(0)$ $(k = 1,2)$. Consequently, $g = (-y_2) - (-y_1)$, and, obviously, $-y_k \in \pi_{G_k}^{-1}(0)$ $(k = 1,2)$, whence, by Theorem 1, there exists an $x \in C([0,1])$ satisfying (29).

Case 3. $-b/a = \frac{1}{2}$. Put

$$y_1 = -g, \qquad y_2 = g + y_1 = 0. \tag{37}$$

Then y_1 satisfies (30) with $t_1 = 0$, $t_2 = 1$ (since $g(0) = b$, $g(1) = a + b = -2b + b = -b$), and y_2 obviously satisfies (31) and (32), whence, by Theorem 1, there exists an $x \in [0,1]$ satisfying (29).

Case 4. $a > 0$, and $\frac{1}{2} < -b/a < 1$. Then $g(1-t) = -at + (a+b)$, with $-a < 0$, and $0 < (a+b)/a = 1 - (-b/a) < \frac{1}{2}$, whence, by case 2 above, $g(1-t) = y_2(t) - y_1(t)$, with $y_k \in \pi_{G_k}^{-1}(0)$ $(k = 1,2)$. Consequently, $g(t) = y_2(1-t) - y_1(1-t)$ $(t \in [0,1])$, and, by (30), (31), $y_k(1-t) \in \pi_{G_k}^{-1}(0)$ $(k = 1,2)$, whence, by Theorem 1, there exists an $x \in C([0,1])$ satisfying (29).

Case 5. $a < 0$, and $\frac{1}{2} < -b/a < 1$. Then $g(1-t) = -at + (a+b)$, with $-a > 0$, and $0 < (a+b)/a = 1 - (-b/a) < \frac{1}{2}$, and we proceed as in case 4, with the only difference that now we use case 1. This completes the proof of Theorem 7.

Remark 6. The above arguments can probably be extended to yield the same result for an arbitrary Čebyšev system z_1, z_2 instead of $z_1(t) \equiv 1$, $z_2(t) \equiv t$. (Recall that a system of n functions, $z_1 \ldots, z_n$, in $C(Q)$ (Q compact) is called a *Čebyšev system* on Q [1], if every nonzero linear combination $\sum_1^n \alpha_i z_i$, has at most $n - 1$ zeros in Q.)

For more than two functions we know only the following necessary condition. Both the theorem and the proof are due to T. J. Rivlin [5].

THEOREM 8. *Let* $E = C([0,1])$, $G_k = [z_1, \ldots, z_k]$, *where* $z_k(t) \equiv t^{k-1}$ $(k = 1, \ldots, n)$ *and* $g_k \in G_k$ $(k = 1, \ldots, n)$. *If there exists an* $x \in E$ *satisfying*

$$\pi_{G_k}(x) = g_k \qquad (k = 1, \ldots, n), \tag{38}$$

then for each pair of indices l, k *with* $1 \leqslant l < k \leqslant n$, *the polynomial* $g_k - g_l$ *is either* $\equiv 0$, *or changes sign at at least* l *distinct points in* $[0,1]$.

Proof. Suppose that $1 \leqslant l < k \leqslant n$, $g_k - g_l \neq 0$. By the alternation theorem of Čebyšev, there exist $l+1$ distinct points, $t_1, \ldots, t_{l+1}$, such that

$$x(t_1) - g_l(t_1) = -[x(t_2) - g_l(t_2)] = \ldots = (-1)^l[x(t_{l+1}) - g_l(t_{l+1})] \\ = \delta\|x - g_l\|, \tag{39}$$

where $\delta = \pm 1$. We claim that

$$\|x - g_k\| < \|x - g_l\|. \tag{40}$$

Indeed, by $G_l \subset G_k$ and (38), we have $\|x - g_k\| \leqslant \|x - g_l\|$. Now, if we had $\|x - g_k\| = \|x - g_l\|$, then, again by $G_l \subset G_k$ and (38), we would have $g_k = \pi_{G_k}(x)$, $g_l = \pi_{G_l}(x)$, contradicting the assumption that $g_k - g_l \neq 0$. This proves (40).

Consequently, by (40) and (39), the polynomial

$$g_k - g_l = (x - g_l) - (x - g_k)$$

has the same sign as $(x - g_l)$ at $t_1, \ldots, t_{l+1}$, whence, by (39), it has at least l sign changes, which completes the proof.

In the case when $n = 2$, the condition of Theorem 8 is also sufficient, as shown by Theorem 7. We do not know whether this condition remains sufficient if $n > 2$.

THEOREM 9. *Let* $E = l_3^\infty =$ *the space of all triplets of real scalars* $x = \{\xi_1, \xi_2, \xi_3\}$, *endowed with the norm* $\|x\| = \max_{1 \leqslant i \leqslant 3}|\xi_i|$ (i.e., $E = C(Q)$, *where* Q *consists of three points*). *Let* G_1, G_2 *be the Čebyšev subspaces* $G_1 = [z_1]$, $G_2 = [z_1, z_2]$, *where* $z_1 = \{1,1,1\}$, $z_2 = \{0, \frac{1}{2}, 1\}$, *and let* $g_k \in G_k$ $(k = 1,2)$. *There exists an* $x \in E$ *satisfying* (29), *if and only if the point* $g = g_1 - g_2 = \{\gamma_1, \gamma_2, \gamma_3\}$ *satisfies either*

$$-3\gamma_3 \leqslant \gamma_1 \leqslant -\tfrac{1}{3}\gamma_3 \tag{41}$$

or

$$-\tfrac{1}{3}\gamma_3 \leqslant \gamma_1 \leqslant -3\gamma_3. \tag{42}$$

Proof. Observe, first, that $g = \{\gamma_1, \gamma_2, \gamma_3\} \in G_2 = [z_1, z_2]$ is of the form $\alpha_1 z_1 + \alpha_2 z_2 = \{\alpha_1, \alpha_1 + (\alpha_2/2), \alpha_1 + \alpha_2\}$, with suitable α_1, α_2; whence

$$\gamma_2 = \tfrac{1}{2}(\gamma_1 + \gamma_3). \tag{43}$$

Assume now that there exists an $x \in E$ satisfying (29). Then, by Theorem 1 and the "alternation theorem" for Čebyšev systems in $C(Q)$ spaces (see e.g. [6], Ch. II, Theorem 1.4), there exist elements

$$y_1 \in \pi_{G_1}^{-1}(0) = \{y = \{\eta_1, \eta_2, \eta_3\} \in E \mid \text{there exist } 1 \leqslant i < j \leqslant 3 \text{ with } \eta_i = -\eta_j = \delta\|y\|\}, \tag{44}$$

$$y_2 \in \pi_{G_2}^{-1}(0) = \{y = \{\eta_1, \eta_2, \eta_3\} \in E \mid \eta_1 = -\eta_2 = \eta_3 = \delta'\|y\|\}, \tag{45}$$

where $\delta, \delta' = \pm 1$, such that

$$g = g_2 - g_1 = y_2 - y_1. \tag{46}$$

Let $y_1 = \{\eta_1{}^1, \eta_2{}^1, \eta_3{}^1\}$. Then, by (46), we have

$$y_2 = \{\gamma_1 + \eta_1{}^1, \gamma_2 + \eta_2{}^1, \gamma_3 + \eta_3{}^1\},$$

whence, by (45),

$$\gamma_1 + \eta_1{}^1 = -\gamma_2 - \eta_2{}^1 = \gamma_3 + \eta_3{}^1. \tag{47}$$

By (44), we have to consider the following three cases:

Case 1. $\eta_1{}^1 = -\eta_2{}^1 = \delta\|y_1\|$. Then, from (47) and (43) we infer $\gamma_1 = -\gamma_2 = (-\gamma_1 - \gamma_3)/2$, whence $\gamma_1 = -\frac{1}{3}\gamma_3$, and thus (41) is satisfied.

Case 2. $-\eta_2{}^1 = \eta_3{}^1 = \delta\|y_1\|$. Then from (47) and (43) we infer $\gamma_3 = -\gamma_2 = (-\gamma_1 - \gamma_3)/2$, whence $\gamma_1 = -3\gamma_3$ and, thus, (41) is satisfied.

Case 3. $\eta_1{}^1 = -\eta_3{}^1 = \delta\|y_1\|$. Then from (47 and) (43) we infer

$$\eta_1{}^1 = \frac{-\gamma_1 + \gamma_3}{2},$$

$$\eta_2{}^1 = -\gamma_2 - \gamma_1 + \frac{\gamma_1 - \gamma_3}{2} = \frac{-2\gamma_2 - \gamma_1 - \gamma_3}{2} = \frac{-2\gamma_1 - 2\gamma_3}{2},$$

whence

$$|2\gamma_1 + 2\gamma_3| = 2|\eta_2{}^1| \leqslant 2\|y_1\| = 2|\eta_1{}^1| = |\gamma_1 - \gamma_3|. \tag{48}$$

Now, if $\gamma_1 - \gamma_3 \geqslant 0$, then (48) implies $\gamma_3 - \gamma_1 \leqslant 2\gamma_1 + 2\gamma_3 \leqslant \gamma_1 - \gamma_3$, whence $-\frac{1}{3}\gamma_3 \leqslant \gamma_1 \leqslant -3\gamma_3$, and, thus, (42) is satisfied. On the other hand, if $\gamma_1 - \gamma_3 \leqslant 0$, then (48) implies $\gamma_1 - \gamma_3 \leqslant 2\gamma_1 + 2\gamma_3 \leqslant \gamma_3 - \gamma_1$, whence $-3\gamma_3 \leqslant \gamma_1 \leqslant -\frac{1}{3}\gamma_3$, and, thus, (41) is satisfied.

Conversely, assume that $g = \{\gamma_1, \gamma_2, \gamma_3\}$ satisfies (41) or (42), whence $|2\gamma_1 + 2\gamma_3| \leqslant |-\gamma_1 + \gamma_3|$. Then, taking $y_1 = \{\eta_1{}^1, \eta_2{}^1, \eta_3{}^1\}$, where

$$\eta_1{}^1 = \frac{-\gamma_1 + \gamma_3}{2}, \qquad \eta_2{}^1 = \frac{-2\gamma_1 - 2\gamma_3}{2}, \qquad \eta_3{}^1 = -\eta_1{}^1, \tag{49}$$

and taking $y_2 = \{\gamma_1 + \eta_1{}^1, \gamma_2 + \eta_2{}^1, \gamma_3 + \eta_3{}^1\}$, by (43) we shall have (44), (45) and (46), whence, by Theorem 1, there exists an $x \in E$ saitsfying (29), which completes the proof of Theorem 9.

Remark 7. If we regard the space $E = l_3{}^\infty$ as $C(Q)$, where $Q = \{0, \frac{1}{2}, 1\}$, and $z_1 = \{1, 1, 1\}$, $z_2 = \{0, \frac{1}{2}, 1\}$ as the restrictions to Q of the functions $\phi_1(t) \equiv 1$, and $\phi_2(t) \equiv t$, respectively, then, by an easy computation, the condition of Theorem 9 is equivalent to the following: g is the restriction to Q of a linear function $\gamma(t) \equiv at + b$ such that $a \neq 0$, $\frac{1}{4} \leqslant -b/a \leqslant \frac{3}{4}$.

We shall say that a pair $\{G_1, G_2\}$ of linear subspaces of a normed linear space E has *property* (A_2), if for every pair $\{g_1, g_2\}$ with $g_k \in G_k$ $(k = 1, 2)$, there exists an $x \in E$ such that

$$g_k \in \mathscr{P}_{G_k}(x) \qquad (k = 1, 2). \tag{50}$$

THEOREM 10. *A pair* $\{G_1, G_2\}$ *of linear subspaces of* $E = l_3^\infty$, *with* $G_1 \subset G_2$, $\dim G_k = k$ $(k = 1, 2)$ *has property* (A_2), *if and only if* G_1 *is a coordinate axis and* G_2 *is a plane passing through* G_1.

Proof. If $\{G_1, G_2\}$ has property (A_2), then, by Theorem 4, both G_1 and G_2 must be non-Čebyšev subspaces, whence, by the classical theorem of Haar, G_1 must be contained in a coordinate plane, and G_2 must be a plane passing through a coordinate axis. Hence we have to consider the following two cases:

Case 1. G_1 is the intersection of G_2 with the coordinate plane perpendicular to the coordinate axis through which G_2 passes. Take $g_1 \in G_1$, and $g_2 \in G_1 \setminus \{g_1\}$. Then a simple computation shows that G_1 is "Čebyšev with respect to the set $\pi_{G_2}^{-1}(g_2)$", i.e., every $x \in \pi_{G_2}^{-1}(g_2)$ has g_2 as unique element of best approximation in G_1:

$$\pi_{G_1}(x) = g_2 \qquad (x \in \pi_{G_2}^{-1}(g_2)). \tag{51}$$

Consequently, there is no $x \in E$ satisfying (50), and, thus, $\{G_1, G_2\}$ does not have property (A_2).

Case 2. G_1 is the coordinate axis through which G_2 passes. Then, again a simple computation shows that $\{G_1, G_2\}$ has property (A_2), which completes the proof of Theorem 10.

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